MA1300 - Enhanced Calculus and Linear Algebra Ⅰ - Notes

Textbook

Chapter 0 Functions

Chapter 1 Limits

1.4 Precise Definition of Limit

	$\lim_{x\to c} = L$	$\lim_{x\to c} \neq L$
Given	$\forall\ \epsilon > 0$	$\exists\ \epsilon > 0$
Given	$\exists\ \delta > 0$	$\forall\ \delta > 0$
Given	$\forall\ 0 < \lvert x-c\rvert < \delta$	$\exists\ 0 < \lvert x-c\rvert < \delta$
Result	$\lvert f(x)-L\rvert < \epsilon$	$\lvert f(x)-L\rvert > \epsilon$

1.9 Continuity at a point (p. 82)

If f is continuous at a,

$$\lim_{x\to a} f(x) = f(a)$$

which implies that:

- $f(a)$ is defined
- $\lim_{x\to a} f(x)$ exists

Self Practice 2

(4) Prove: $\lim_{x\to 0} \sqrt{x^3+x^2}\sin\frac{\pi}{x} = 0$

> $$\sqrt{x^3+x^2} = |x|\sqrt{1+x}$$
> $$\underline{-\sqrt{\frac{1}{2}}|x|} \le -\sqrt{x^3+x^2} \le \sqrt{x^3+x^2}\sin\frac{\pi}{x} \le \sqrt{x^3+x^2} \le \underline{\sqrt{\frac{3}{2}}|x|}$$
> $$\text{where } 0 < |x| < \frac{1}{2}$$
> $$\lim_{x\to 0}\left(-\sqrt{\frac{1}{2}}|x|\right) = \lim_{x\to 0}\left(\sqrt{\frac{3}{2}}|x|\right) = 0$$

(12) Prove: $\lim_{x\to 0} f(x) = 0$, where

$$f(x) = \begin{cases} x^2 & x\in\mathbb{Q} \\ 0 & x\notin\mathbb{Q} \end{cases}$$

> $$0 \le f(x) \le x^2$$
> $$\lim_{x\to0} 0 = \lim_{x\to0} x^2 = 0$$

(13) Evaluate: $\lim_{x\to2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$

> $$\begin{aligned}\lim_{x\to2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} &= \lim_{x\to2} \frac{(6-x-4)\left(\sqrt{3-x}+1\right)}{(3-x-1)\left(\sqrt{6-x}+2\right)} \\ &= \lim_{x\to2} \frac{(2-x)\left(\sqrt{3-x}+1\right)}{(2-x)\left(\sqrt{6-x}+2\right)} \\ &= \lim_{x\to2} \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{1}{2}\end{aligned}$$

(15) Prove: $\lim_{x\to0} f(x)$ DNE, where

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

Solution. Suppose $\lim_{x\to0} f(x) = \Delta$, Let $\varepsilon = 1/4$, then for any $\delta > 0$, there exists a rational number $0 < x_\delta < \delta$, and an irrational number $0 < x'_\delta < \delta$, so $f(x_\delta) = 0$, $f(x'_\delta) = 1$, and $\varepsilon < 1 \le |f(x_\delta) - \Delta| + |f(x'_\delta) - \Delta|$. Therefore $\lim_{x\to0} f(x)$ does not exist.

> Template for proving limit is not f_0:
>
> $$\exists\, \epsilon > 0, \forall\, \delta, \exists\, 0 < |x_1 - x_0| < \delta \Rightarrow |f(x_1) - f_0| \ge 0$$
>
> Template for proving limit DNE:
>
> $n \to \infty, a_n \to x_0 \Rightarrow f(a_n) \to \lim_{x\to x_0} f(x)$
>
> $$\{a_n\} \to x_0, \{b_n\} \to x_0, \lim_{n\to\infty} f(a_n) \ne \lim_{n\to\infty} f(b_n)$$
>
> More precisely for this question:
>
> $$\begin{aligned} n \to \infty, \mathbf{A_n} = \frac{\mathbf{1}}{\mathbf{n}} &\to \mathbf{0}, \mathbf{B_n} = \frac{\sqrt{\mathbf{2}}}{\mathbf{n}} \to \mathbf{0} \\ \exists\, \epsilon = \frac{1}{2}, \forall\, \delta, \exists\, (n+1) &= \lceil \frac{\sqrt{2}}{\delta} \rceil + 1 \\ B_{n+1} = \frac{\sqrt{2}}{n+1} = \frac{\sqrt{2}}{\lceil \frac{\sqrt{2}}{\delta} \rceil + 1} &< \frac{\sqrt{2}}{\frac{\sqrt{2}}{\delta}} = \delta \\ |f(B_{n+1}) - 0| &= 1 > \epsilon \\ \therefore \lim_{x\to0} f(x) &\ne 0 \end{aligned}$$
>
> $$\begin{aligned} \exists\, \epsilon = \frac{1}{2}, \forall\, \delta, \exists\, (n+1) &= \lceil \frac{1}{\delta} \rceil + 1 \\ A_{n+1} = \frac{1}{n+1} = \frac{1}{\lceil \frac{1}{\delta} \rceil + 1} &< \frac{1}{\frac{1}{\delta}} = \delta \\ |f(A_{n+1}) - 1| &= 0 > \epsilon \\ \therefore \lim_{x\to0} f(x) &\ne 1 \end{aligned}$$

Self Practice 3

Use the Definition of Continuity

(1, 2)

> - **Since f is a rational function**
> - **and $a = 1$ is in domain/$(1, \infty)$is a subset of domain**
> - so $\lim\limits_{x \to 1} f(x) = f(1)$

Use Continuity to Evaluate Limit

(5)

> - Example: $\lim\limits_{x \to \pi} \sin(x + \sin x)$
> - Since $\sin t$ domain is $\mathbb{R}$, $\lim\limits_{x \to \pi}(x + \sin x) = \pi \in \mathbb{R}$,
> - So $\lim\limits_{x \to \pi} \sin(x + \sin x) = \sin\left(\lim\limits_{x \to \pi}(x + \sin x)\right) = \sin \pi = 0$

Self Practice 4

(4)

If a and b are positive numbers, prove $\frac{a}{x^3+2x^2-1} + \frac{b}{x^3+x-2} = 0$ have at least one solution on the interval $(-1, 1)$.

$$
\begin{aligned}
f(x) &= x^3 + 2x^2 - 1, f(-1) = 0 \\
f(x) &= (x^3 + x^2) + (x^2 + x) - (x + 1) = (x^2 + x - 1)(x + 1) \\
g(x) &= x^3 + x - 2, g(1) = 0 \\
g(x) &= (x^3 - x^2) + (x^2 - x) + (2x - 2) = (x^2 + x + 2)(x - 1) \\
\Rightarrow H(x) &= \frac{a}{(x+1)(x - \frac{-1-\sqrt{5}}{2})(x - \frac{-1+\sqrt{5}}{2})} + \frac{b}{(x-1)(x^2+x+2)} \\
x &\in (-1, \phi) \cup (\phi, 1), \phi = \frac{\sqrt{5}-1}{2} \\
\lim_{x \to \phi^+} &H(x) = \infty + C = +\infty, \lim_{x \to 1^-} H(x) = C - \infty = -\infty \\
&\textbf{As H(x) is continuous on } (\phi, 1), \\
&\exists\, \delta_1 > 0, \forall\ 0 < \hat{\delta}_1 < \delta_1, H(1 - \hat{\delta}_1) < -1 \\
&\exists\, \delta_2 > 0, \forall\ 0 < \hat{\delta}_2 < \delta_2, H(\phi + \hat{\delta}_2) > 1 \\
&\textbf{Let } \delta = \min\left\{\frac{\delta_1}{2}, \frac{\delta_2}{2}, \frac{1-\phi}{3}\right\} \\
&\begin{cases} H(1-\delta) < -1 \\ H(\phi + \delta) > 1 \\ \quad H(x) \textbf{ continuous on } (\phi + \delta, 1 - \delta) \end{cases} \\
\Rightarrow H(x) &= 0 \textbf{ have at least one solution on } (\phi + \delta, 1 - \delta)
\end{aligned}
$$

Chapter 2 Derivatives

2.2 The derivative as a function

Ex. if $f(x)=\sqrt{x}$, find $f'(x)$ with its domain.

> $$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0}\frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\ &= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}(x>0) \end{aligned}$$
>
> $f(x)$ is not differentiable at $x=0$, because it corners at $x=0$.

Ex. Prove $\frac{\mathrm{d}x^n}{\mathrm{d}x}=nx^{n-1}$.

> $$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{-x^n+(x+h)^n}{h} \\ &= \lim_{h\to 0}\frac{-x^n+x^n+\binom{n}{1}x^{n-1}h^1+\cdots+\binom{n}{k}x^{n-k}h^k+\cdots+h^n}{h} \\ &= \lim_{h\to 0}\frac{\binom{n}{1}x^{n-1}h+\dots}{h} \\ &= \binom{n}{1}x^{n-1}=nx^{n-1} \end{aligned}$$

2.3 Differentiation formulas

Proof of Product Rule

> $$\begin{aligned} [f(x)g(x)]' &= \lim_{h\to 0}\frac{-f(x)g(x)+f(x+h)g(x+h)}{h} \\ &= \lim_{h\to 0}\frac{-f(x)g(x)+[f(x)+f'(x)h][g(x)+g'(x)h]}{h} \\ &= \lim_{h\to 0}\frac{-f(x)g(x)+f(x)g(x)+f'(x)g(x)h+f(x)g'(x)h+f'(x)g'(x)h^2}{h} \\ &= \lim_{h\to 0}f'(x)g(x)+f(x)g'(x)+f'(x)g'(x)h \\ &= f'(x)g(x)+f(x)g'(x) \end{aligned}$$

Proof of Quotient Rule

> $$\begin{aligned} \left[\frac{f(x)}{g(x)}\right]' &= \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x)g(x+h)h} \\ &= \lim_{h\to 0}\frac{[f(x)+f'(x)h]g(x)-f(x)[g(x)+g'(x)h]}{g(x)[g(x)+g'(x)h]\cdot h} \\ &= \lim_{h\to 0}\frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)+g(x)g'(x)h} \\ &= \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)} \end{aligned}$$

2.4 Derivatives of trigonometric functions

Proof of Limit formula (2)

$$\begin{aligned}
\because \cos 2\theta &= 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \\
\therefore \lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} &= \lim_{\theta\to 0}\frac{-2\sin^2\frac{\theta}{2}}{\theta} \\
&= -\lim_{\theta\to 0}\frac{\sin^2\frac{\theta}{2}}{\frac{\theta}{2}} \\
&= -\lim_{\theta\to 0}\frac{\sin^2\frac{\theta}{2}}{\left(\frac{\theta}{2}\right)^2}\lim_{\theta\to 0}\frac{\theta}{2} \\
&= -\lim_{\theta\to 0}\frac{\theta}{2} = 0 \\
\textbf{another proof} & \\
\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} &= \lim_{\theta\to 0}\frac{(\cos\theta - 1)(\cos\theta + 1)}{\theta(\cos\theta + 1)} \\
&= \lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos\theta + 1)} \\
&= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta + 1} \\
&= -1\times\frac{0}{1+1} = 0
\end{aligned}$$

Proof of Derivatives

$$\begin{aligned}
(\tan x)' &= \frac{\sin x'\cos x - \sin x\cos x'}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}$$

$$\begin{aligned}
(\cos x)' &= \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h} \\
&= \cos x\lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x\lim_{h\to 0}\frac{\sin h}{h} \\
&= \cos x\cdot 0 - \sin x\cdot 1 \\
&= -\sin x
\end{aligned}$$

- $(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\csc^2 x$
- $(\sec x)' = [(\cos x)^{-1}]' = \frac{-\sin x}{-(\cos x)^2} = \tan x\sec x$
- $(\csc x)' = [(\sin x)^{-1}]' = \frac{\cos x}{-(\sin x)^2} = -\cot x\csc x$

2.6 Implicit Differentiation

Definition

Treat y as a function of x.

$$\frac{\mathrm{d}}{\mathrm{d}x} y = y'$$
$$\therefore \frac{\mathrm{d}}{\mathrm{d}x} y^2 = 2yy'$$

Example

Find the tangent line of the equation $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ at $(3, 1)$.

> $$\begin{aligned} 2\frac{d}{dx}[(x^2+y^2)^2] &= 25\left[\frac{d}{dx}x^2 - \frac{d}{dx}y^2\right] \\ 2 \cdot 2(x^2+y^2) \cdot \frac{d}{dx}(x^2+y^2) &= 25\left(2x - \frac{d}{dx}y^2\right) \\ 4(x^2+y^2)\left(2x + \frac{d}{dx}y^2\right) &= 50x - 25[2y \cdot \frac{d}{dx}y] \\ 4(x^2+y^2)(2x+2yy') &= 50(x - yy') \\ y' &= \frac{25x - 4x^3 - 4xy^2}{4x^2y + 4y^3 + 25y} \\ y'\big|_{x=3} &= -\frac{9}{13} \end{aligned}$$

Example

- Car A is traveling west at 40 km/h and car B is traveling north at 50 km/h.
- Both are headed for the intersection of the two roads.
- At what rate are the cars approaching each other when car A is 0.6 km and car B is 0.4 km from the intersection?

> $$\begin{aligned} \frac{dx}{dt} &= 40 \\ \frac{dy}{dt} &= 50 \\ D &= \sqrt{x^2+y^2} \\ \frac{dD}{dt} &= \frac{d}{dt}\sqrt{x^2+y^2} \\ &= \frac{d}{dt}(x^2+y^2) \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \\ &= \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2\sqrt{x^2+y^2}} \\ &= \frac{2(0.6 \cdot 40 + 0.4 \cdot 50)}{2\sqrt{0.6^2+0.4^2}} \end{aligned}$$

V EXAMPLE 5 A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

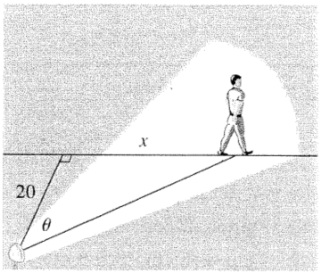

FIGURE 5

SOLUTION We draw Figure 5 and let x be the distance from the man to the point on the path closest to the searchlight. We let θ be the angle between the beam of the searchlight and the perpendicular to the path.

We are given that $dx/dt = 4$ ft/s and are asked to find $d\theta/dt$ when $x = 15$. The equation that relates x and θ can be written from Figure 5:

$$\frac{x}{20} = \tan\theta \qquad x = 20\tan\theta$$

Differentiating each side with respect to t, we get

$$\frac{dx}{dt} = 20\sec^2\theta\,\frac{d\theta}{dt}$$

so

$$\frac{d\theta}{dt} = \frac{1}{20}\cos^2\theta\,\frac{dx}{dt} = \frac{1}{20}\cos^2\theta\,(4) = \frac{1}{5}\cos^2\theta$$

When $x = 15$ ft, the length of the beam is 25 ft, so $\cos\theta = \frac{4}{5}$ and

$$\frac{d\theta}{dt} = \frac{1}{5}\left(\frac{4}{5}\right)^2 = \frac{16}{125} = 0.128$$

The searchlight is rotating at a rate of 0.128 rad/s. ■

> **Important!** $\frac{d\tan\theta}{dt} = \frac{1}{\cos^2\theta}\frac{d\theta}{dt}$

Aware of $f(0)$ Case

Webwork 2.1 (3)

$$f(x) = \begin{cases} -5x^2 + 2x & x < 0 \\ 8x^2 - 3 & x \geq 0 \end{cases}$$

> According to the definition of the derivate, to compute $f'(0)$,
>
> - $\lim\limits_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^-} \frac{-5x^2+2x-3}{x} = \lim\limits_{x\to 0^-} -5x + 2 + \frac{3}{x}$ DNE
> - $\lim\limits_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^+} \frac{8x^2-3-(-3)}{x} = \lim\limits_{x\to 0^+} 8x = 0$
> - So $f'(0)$ is undefined.

L' hospitals Case

Webwork 2.3 (4) Find $\lim\limits_{x\to 0} \frac{\cot 5x}{\csc x}$

> $$\begin{aligned} \lim_{x\to 0} \frac{\cot 5x}{\csc x}\left(\frac{\infty}{\infty}\right) &= \lim_{x\to 0} \cot 5x \sin x \\ &= \lim_{x\to 0} \frac{\sin x}{\tan 5x}\left(\frac{0}{0}\right) \\ &= \lim_{x\to 0} \frac{\cos x}{\frac{5}{\cos^2 5x}} \\ &= \lim_{x\to 0} \frac{1}{5} = \frac{1}{5} \end{aligned}$$

[Important] $\frac{\mathrm{d}}{\mathrm{d}x}|x| = \frac{x}{|x|} = \operatorname{sgn} x$

> Proof: $|x|' = \left(\sqrt{x^2}\right)' = \frac{1}{2}(x^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}$
>
> Example: (Webwork 2.4 (3))
>
> - $|\sin x|' = \frac{\sin x}{|\sin x|} \cdot \cos x$
> - $\sin |x|' = \frac{x}{|x|} \cdot \cos |x| = \frac{x}{|x|} \cdot \cos x$

Double-Implicit Differenation

Self Practice 8 (5) $l \sin \alpha = 50$, $\frac{d(l\cos\alpha)}{dt} = 2$, Solve $\frac{d\alpha}{dt}$ when $l = 100$

> $$\begin{aligned} l \sin \alpha &= 50 \\ \Rightarrow l &= \frac{50}{\sin \alpha} \\ \frac{d(l\cos\alpha)}{dt} &= 2 \\ \Rightarrow \frac{d(50\cot\alpha)}{dt} &= 2 \\ \frac{d\cos\alpha}{dt} &= -\frac{1}{\sin^2\alpha}\color{red}{\frac{d\alpha}{dt}} = \frac{2}{50} \\ l = 100 \Rightarrow \sin\alpha &= \frac{\pi}{6} \\ \left.\frac{d\alpha}{dt}\right|_{l=100} &= -\frac{1}{25}\sin^2\alpha = -\frac{1}{100} \end{aligned}$$

Midterm

Limit Proof by Definition

Midterm 2(a) Prove $\lim\limits_{x\to 0} \frac{x-2}{x^2+x+1} = -2$

> $$\begin{aligned} &\forall\, \epsilon > 0 \\ &\exists\, \delta = \min\left\{1, \frac{3\epsilon}{20}\right\} \\ &\forall\, 0 < |x| < \delta \\ |f(x) - (-2)| &= \left|\frac{2x^2+3x}{x^2+x+1}\right| \\ &= |x|\left|\frac{3+2x}{(x+\frac{1}{2})^2+\frac{3}{4}}\right| \\ &\le |x|\left|\frac{3+2x}{\frac{3}{4}}\right| \\ &\le |x|\left|\frac{3+2}{\frac{3}{4}}\right| (\delta \le 1) \\ &= \frac{20}{3}|x| \\ &< \frac{20}{3}|\delta| < \epsilon \end{aligned}$$

Limit DNE Proof by Definition

Midterm 2(b) Prove $\lim_{x\to 0^-}\frac{\sin\frac{1}{x^2}}{x^2}$ DNE

> $$\begin{aligned}\textbf{Suppose } \lim_{x\to 0^-}\frac{\sin\frac{1}{x^2}}{x^2} &= L\\ \textbf{by Definiton, Let } \epsilon &= 1\\ \forall\,\delta &> 0\\ \forall-\delta < x &< 0\\ \rightarrow\left|\frac{\sin\frac{1}{x^2}}{x^2}-L\right| &< 1\\ \textbf{Let } x_0 &= -\frac{1}{\sqrt{2n\pi+\frac{\pi}{2}}}\\ -\delta < x_0 &< 0\\ \textbf{However}\left|\frac{\sin\frac{1}{x_0^2}}{x_0^2}-L\right| &= \left|2n\pi+\frac{\pi}{2}-L\right| > 1\end{aligned}$$
>
> Between each two $f(x_0)=2n\pi+\frac{\pi}{2}$, there is a gap of $2\pi > 2$, thus there is no L such that all $|f(x_0)-L|\le 1$.

MVT

Midterm 4 $f(x)$ continuous on $[0,n]$, $f(0)=f(n)$, Prove: $\exists\, x\in[0,n-1]$, $f(x)=f(x+1)$.

> $$\begin{aligned}&\textbf{Let } g(i)=f(i)-f(i+1)\ (0\le i\le n-1)\\ &\quad\textbf{Case 1}\\ &\qquad g(0)=g(1)=\cdots=g(n-1)=0\\ &\quad\textbf{Case 2}\\ &\qquad \exists\, g(a)>0, g(b)<0\\ &\textbf{According to MVT,}\\ &\qquad \exists\,\zeta\in(a,b)\subset[0,n-1]\\ &\qquad g(\zeta)=f(\zeta)-f(\zeta+1)=0\end{aligned}$$

Squeeze Theorem

> $$\begin{aligned}-h \le h\sin\frac{1}{h} &\le h\\ \lim_{x\to 0}-h = \lim_{x\to 0}h &= 0\\ \Rightarrow \lim_{x\to 0}h\sin\frac{1}{h} &= 0\end{aligned}$$

Chapter 4 Inverse Functions

Log Diff Trick

(Ex on Page 6) Find y' for $y=\frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

> $$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$
> $$\frac{dy}{dx}\frac{1}{y} = \frac{3}{4x} + \frac{2x}{2(x^2+1)} - \frac{15}{3x+2}$$
> $$y' = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Inverse Trigonometric Function

Inv Function	Range	Domain	SP 1	SP 2	SP 3	SP 4
$y=\sin^{-1}x$	$\lvert x\rvert<1$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	$-1,-\frac{\pi}{2}$	$0,0$	$1,\frac{\pi}{2}$	
$y=\cos^{-1}x$	$\lvert x\rvert<1$	$(0,\pi)$	$1,0$	$0,\frac{\pi}{2}$	$-1,\pi$	
$y=\tan^{-1}x$		$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	$-\infty,-\frac{\pi}{2}$	$0,0$	$\infty,\frac{\pi}{2}$	
$y=\cot^{-1}x$		$(0,\pi)$	$\infty,0$	$0,\frac{\pi}{2}$	$-\infty,\pi$	
$y=\sec^{-1}x$	$\lvert x\rvert\geq 1$	$\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$	$1,0$	$\infty,\frac{\pi}{2}$	$-1,\pi$	$-\infty,\frac{3\pi}{2}$
$y=\csc^{-1}x$	$\lvert x\rvert\geq 1$	$\left(0,\frac{\pi}{2}\right]\cup\left(\pi,\frac{3\pi}{2}\right]$	$\infty,0$	$1,\frac{\pi}{2}$	$-\infty,\pi$	$-1,\frac{3\pi}{2}$

Notice: $x=\pm 1$ for $\sin^{-1}, \cos^{-1}$ and $x=\pm\infty$ for all cases are **not in the range**

Inv Trig Derivative

Assume $\theta = \text{func}^{-1}\, x$ in all cases.

Important: $(1+\tan^2\theta)\cos^2\theta = \cos^2\theta + \sin^2\theta = 1 \Leftrightarrow (1+\cot^2\theta)\sin^2\theta = 1$

$\Rightarrow \tan^2\theta = \sec^2\theta - 1, \cot^2\theta = \csc^2\theta - 1$

- $\frac{d}{dx}\sin^{-1}x = \frac{d\theta}{d\sin\theta} = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\cos^{-1}x = \frac{d\theta}{d\cos\theta} = \frac{-1}{\sin\theta} = \frac{-1}{\sqrt{1-\cos^2\theta}} = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\tan^{-1}x = \frac{d\theta}{d\tan\theta} = \frac{1}{1/\cos^2\theta} = \frac{1}{1+\tan^2\theta} = \frac{1}{1+x^2}$
- $\frac{d}{dx}\cot^{-1}x = \frac{d\theta}{d\cot\theta} = \frac{-1}{1/\sin^2\theta} = \frac{-1}{1+\cot^2\theta} = \frac{-1}{1+x^2}$
- $\frac{d}{dx}\sec^{-1}x = \frac{d\theta}{d\sec\theta} = \frac{1}{\tan\theta\sec\theta} = \frac{1}{\sqrt{\sec^2\theta-1}\sec\theta} = \frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}\csc^{-1}x = \frac{d\theta}{d\csc\theta} = \frac{-1}{\cot\theta\csc\theta} = \frac{-1}{\sqrt{\csc^2\theta-1}\csc\theta} = \frac{-1}{x\sqrt{x^2-1}}$

L' hospital's Rule

Type 1^∞

(Ex on Page 11)

$$
\begin{aligned}
\lim_{x\to0^+}(1+\sin 4x)^{\cot x}(1^\infty) &= \lim_{x\to0^+}\left(e^{\ln(1+\sin 4x)}\right)^{\cot x} \\
&= \lim_{x\to0^+} e^{\ln(1+\sin 4x)\cot x}(e^{0\cdot\infty}) \\
&= \exp \lim_{x\to0^+} \frac{\ln(1+\sin 4x)}{\tan x}\left(\frac{0}{0}\right) \\
&= \exp \lim_{x\to0^+} \frac{\frac{4\cos 4x}{1+\sin 4x}}{\frac{1}{\cos^2 x}} \\
&= \exp \frac{\frac{4\times1}{1+0}}{1} = e^4
\end{aligned}
$$